

Asymptotic behavior of dynamical systems and cellular automata

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Abstract. We study discrete dynamical systems through the topological concepts of limit set, which consists of all points that can be reached arbitrarily late, and asymptotic set, which consists of all adhering values of orbits. In particular, we deal with the case when each of these are a singleton, or when the restriction of the system is periodic on them, and show that this is equivalent to some simple dynamics in the case of subshifts or cellular automata. Moreover, we deal with the stability of these properties with respect to some simulation notions.

Introduction

Complex systems are made of a great number of entities interacting locally with each other in a fully deterministic way. However, in many cases, the global behavior of these systems is very complex and the only known way to understand their dynamics is by simulating them. These systems appear in many different fields such as biology, physics, chemistry, sociology, ...

In order to achieve links between the well-known local rule and the long-term dynamical properties of the system, one of the classical points of view is the study of attractors (see [1]). To achieve formal results on those kinds of properties, some regular model is needed. Therefore, one often introduces some compact topology and some continuous self-map representing the evolution: discrete-time dynamical systems. Besides, if we require spatial homogeneity, we define the model of cellular automata (see [2, 3]), which are composed of an infinite number of cells disposed on a line, and endowed with a state chosen among a finite alphabet.

We focus here on the long-term behavior of such formal systems. It can be first represented by the limit set, which consists of all the configurations that can appear after an arbitrarily long time (see [4, 5]). A more restrictive notion is the asymptotic set, composed of all the configurations close to which the system is passing infinitely often (see [6, 7]).

In this paper, we shall develop a selective review on newly achieved properties linking local behavior and properties of the limit or asymptotic set for dynamical systems in general, cellular automata or subshifts. The paper is divided as follows: after giving all necessary definitions about dynamical systems in Section 1, we shall first treat the case of limit set (Section 2) then the case of asymptotic set (Section 3).

1. Topological dynamics

We will note $\mathbb{N}_+ = \mathbb{N} \setminus \{0\}$ and $\mathbb{2} = \{0, 1\}$.

1.1. Dynamical systems

We model complex systems by discrete (topological) dynamical systems. Even though “real life” time is continuous and making it discrete can introduce artifacts, it can be observed that this restriction already exhibits a very complex behavior.

Definition 1. A discrete dynamical system (DDS) is a pair (X, F) (or simply F when there is no confusion), where X is a nonempty compact metric space and $F : X \rightarrow X$ a continuous function.

Let us denote d the distance on X and $\mathcal{B}_\varepsilon(x) = \{y \in X \mid d(x, y) < \varepsilon\}$ the open ball of center $x \in X$ and radius $\varepsilon > 0$.

A subset $Y \subseteq X$ is F -invariant (resp. strongly F -invariant) if $F(Y) \subseteq Y$ (resp. $F(Y) = Y$). Given such a set, we can define the restriction $F|_Y$ of our dynamical

system to it ; $(Y, F|_Y)$ (or simply (Y, F)) is a *subsystem* of (X, F) if besides Y is closed. A DDS (X, F) is *minimal* if it does not contain any strict subsystem, *i.e.* if any closed F -invariant $Y \subset X$ is either \emptyset or X .

Simulation Let us introduce some order on the dynamics produced by these systems: the simulation. Intuitively, one says that one system simulates another one if the latter can be embedded into the former in a continuous way.

Definition 2. A morphism of a DDS (X, F) into another (Y, G) is a continuous function $\Phi : X \rightarrow Y$ such that $\Phi F = G\Phi$. If the morphism is surjective, it is a factor map, (Y, G) is called factor of (X, F) and (X, F) extension of (Y, G) . If the morphism is bijective, it is a conjugacy; (X, F) and (Y, G) are said conjugate.

A simulation of period $n \in \mathbb{N}_+$ by steps of $n' \in \mathbb{N}_+$ by a DDS (X, F) of another (Y, G) is a factor map of some subsystem (X', F^n) of (X, F^n) into $(Y, G^{n'})$.

Moreover, the simulation is *direct* if $n = 1$, *total* if $n' = 1$, *complete* if $X' = X$, *exact* if the factor map is actually a conjugacy. We say in these cases that (X, F) *simulates directly* (resp. *totally, completely, exactly*) (Y, G) .

$$\begin{array}{ccccc}
 X & \supseteq & X' & \xrightarrow{\quad} & Y \\
 \downarrow F & & \downarrow F & \Phi & \downarrow G \\
 X & & X' & & Y \\
 \vdots & & \vdots & & \vdots \\
 X & & X' & & Y \\
 \downarrow F & & \downarrow F & & \downarrow G \\
 X & \supseteq & X' & \xrightarrow{\quad} & Y \\
 & & & \Phi &
 \end{array}$$

Figure 1. Simulation (Φ is surjective)

Configurations In the rest of the paper, we shall focus on a specific case of dynamical systems: totally disconnected discrete dynamical systems (TDDS). Intuitively, they correspond to discretizing the space where the interacting basic objects live. From now on, A will be some finite alphabet and \mathbb{M} will stand either for \mathbb{N} or for \mathbb{Z} . In this context, a *configuration* of the space is an infinite sequence $x \in A^{\mathbb{M}}$ of letters.

The set $A^{\mathbb{M}}$ is endowed with the topology induced by the distance $d(x, y) = 2^{\min_{x_i \neq y_i} |i|}$, which corresponds to the product topology of the discrete topology on A . From Tychonoff's theorem, the resulting space is compact. It is known that totally disconnect sets are homeomorphic to subsets of $A^{\mathbb{M}}$; hence we will restrict our study of TDDS to the systems (Σ, F) on a subspace $\Sigma \subset A^{\mathbb{M}}$ of configurations.

If $x \in A^{\mathbb{M}}$ is a configuration and $i, k \in \mathbb{M}$, we denote $x_{[i, k]} \sqsubset x$ the finite pattern $x_i x_{i+1} \dots x_{k-1}$. The same notation holds for any kind of interval (including infinite ones). If $u \in A^*$ and $i \in \mathbb{M}$, then $[u]_i$ denotes the *cylinder* $\{x \in A^{\mathbb{M}} \mid x_{[i, i+|u|]} = u\}$. If $k \in \mathbb{N}$, we note $\langle k \rangle = \{i \in \mathbb{M} \mid |i| \leq k\}$. For instance, $A^{\langle k \rangle}$ denotes the set of patterns

of length $k + 1$ if $\mathbb{M} = \mathbb{N}$, of length $2k + 1$ (indexed from $-k$ to k) if $\mathbb{M} = \mathbb{Z}$. If $u \in A^{(k)}$, then $[u]$ denotes the *central cylinder* $\{x \in A^{\mathbb{M}} \mid x_{(k)} = u\}$ of configurations that share u as a central pattern. More generally, if $U \subset A^{(k)}$, the $[U]$ denotes the cylinder $\{x \in A^{\mathbb{M}} \mid x_{(k)} \in U\}$. We can also extend the cylinder notation in the following flavor: $x = {}^\infty 0[u] 0^\infty$ stands for the configuration filled entirely with 0 except for the portion $x_{(k)} = u$; similarly, ${}^\infty 0[u]_i = \{x \in [u]_i \mid \forall j < i, x_j = 0\}$, and so on.

Given two configurations $x, y \in A^{\mathbb{Z}}$ and some cell $i \in \mathbb{M}$, we define the *concatenation* $x \oplus_i y$ as the configuration z such that $z_k = x_k$ if $k < i$, y_k if $k \geq i$.

If $0 \in A$, we say that a configuration $c \in A^{\mathbb{M}}$ is *0-finite* if it is equal to 0 except for some finite number of elements. One easy remark is that the set of 0-finite configurations is dense in $A^{\mathbb{M}}$.

Subshifts A natural operation on configurations consists in “translating” it; this operation is called the *shift* $\sigma : A^{\mathbb{M}} \rightarrow A^{\mathbb{M}}$, defined as $\sigma(x)_i = x_{i+1}$. Most of the time, we choose to study TDDS which are homogeneous and therefore commute with the shift.

Definition 3. A onesided (resp. twosided) subshift is a closed σ -invariant (resp. strongly) subset Σ of $A^{\mathbb{N}}$ (resp. $A^{\mathbb{Z}}$).

Here, we introduce two classical different characterizations using either languages or graphs.

If $L \subset A^*$ is a language, then the set $\Sigma_L = \{z \in A^{\mathbb{M}} \mid \forall u \in L, u \not\sqsubset z\}$ of configurations avoiding patterns of L is a subshift. Conversely, to any subshift Σ can be associated the *language* $\mathcal{L}(\Sigma) = \{u \in A^* \mid \exists z \in \Sigma, u \sqsubset z\}$ of the finite patterns appearing in some of its configurations. For any length $k \in \mathbb{N}$, the *language of order k* of Σ is $\mathcal{L}_k(\Sigma) = \mathcal{L}(\Sigma) \cap A^k$. A *forbidden language* of a subshift Σ is a language $L \subset A^*$ such that $\Sigma = \Sigma_L$. A subshift is *of finite type* (SFT) if it admits some finite forbidden language. It is of *order $k \in \mathbb{N}$* (k -SFT) if it admits a forbidden language included in A^k .

Let us define a *graph* on alphabet A as a pair $\mathcal{G} = (V, E)$ where V is the finite set of *vertices*, $E \subset V \times V \times A$ the finite set of *arcs*; if $(v, w, a) \in E$ then v is the *initial* vertex of the arc, w its *terminal* vertex and a its *label*. A *path* is a sequence $(v_j, w_j, a_j)_{j \in I} \in E^{\mathbb{M}}$ of arcs where $I \subset \mathbb{M}$ and $v_{j+1} = w_j$ for $j, j+1 \in I$. Its *label* is the sequence $(a_j)_{j \in I}$. A graph is *strongly connected* if any two vertices $v, w \in V$ belong to a same path. The *label system* of a graph $\mathcal{G} = (V, E)$ on A is the subshift $\Gamma_{\mathcal{G}} = \{(a_j)_{j \in \mathbb{M}} \mid (v_j, w_j, a_j)_{j \in \mathbb{M}} \in \Sigma_{\mathcal{G}}\}$ of the labels of its infinite paths. A subshift is *sofic* if it is the label system of some graph. If we see the graph as a finite automaton, we can see that a subshift is sofic if and only if its language is regular. Another equivalence is given by the Weiss’ theorem [8]: a subshift is sofic if and only if it is the factor of some SFT.

Cellular automata We introduce a kind of dynamical system which represent spatially homogeneous dynamics: cellular automata. Formally, a (one-dimensional) *cellular automaton* (CA) on alphabet A is a triplet (m, d, f) where $m \in \mathbb{M}$ is the *anchor*,

$d \in \mathbb{N}$ is the *diameter* and $f : A^d \rightarrow A$ the *local transition rule*. We shall assimilate the cellular automaton with its associated dynamical system $(A^{\mathbb{M}}, F)$ defined by $F(x)_i = f(x_{\llbracket i-m, i-m+d \rrbracket})$ for any $x \in A^{\mathbb{M}}$ and any $i \in \mathbb{M}$, under the motivation of the following result.

Theorem 1 (Curtis, Hedlund & Lyndon [3]). *Cellular automata are exactly the TDDS $(A^{\mathbb{M}}, F)$ such that F commutes with the shift σ .*

More generally, we define a *partial cellular automata* (PCA) as a TDDS (Σ, F) where $\Sigma \subseteq A^{\mathbb{M}}$ and F commutes with the shift. One can note that Hedlund's theorem still applies and that such systems correspond to restrictions of cellular automata to subshifts. Therefore, they are also defined thanks to an anchor m , a diameter d and a local function $f : \mathcal{L}_d(\Sigma) \rightarrow A$.

We say that the PCA is *oneway* (resp. *oblic*) if its anchor can be taken $m \leq 0$ or $m \geq d - 1$ (resp. $m < 0$ or $m \geq d$). A state $0 \in A$ is *quiescent* for a PCA (Σ, F) if $0^d \in \mathcal{L}(\Sigma)$ and $f(0, \dots, 0) = 0$. The global rule F of a PCA can be canonically extended to all words by:

$$\begin{aligned} F : \mathcal{L}(\Sigma) &\rightarrow \mathcal{L}(\Sigma) \\ u &\mapsto (f(u_{\llbracket i, i+d \rrbracket}))_{0 \leq i < |u| - d - 1} . \end{aligned}$$

Without loss of generality, we will sometimes assume the neighborhood to be symmetrical, *i.e.* $d = 2m + 1$ if $\mathbb{M} = \mathbb{Z}$, $m = 0$ if $\mathbb{M} = \mathbb{N}$; in that case $r = d - m - 1$ is called the *radius* of the CA.

A basic example of CA that we will use throughout the paper is $\text{Min} : 2^{\mathbb{N}} \rightarrow 2^{\mathbb{N}}$, defined by anchor 0, diameter 2 and local rule:

$$\begin{aligned} f : 2^2 &\rightarrow 2 \\ (a, b) &\mapsto a \times b . \end{aligned}$$

One easy remark is that this CA admits both 0 and 1 as quiescent states.

1.2. Dynamical properties

We now study the dynamics of the previously introduced systems (X, F) , *i.e.* the structure of the *orbits* $\mathcal{O}_F(x) = \{F^t(x) \mid t \in \mathbb{N}\}$ of the points $x \in X$. We note the *positive orbit* $\mathcal{O}_F^+(x) = \{F^t(x) \mid t \in \mathbb{N}_+\}$ of $x \in X$. In the case of a TDDS F , we can depict such an orbit $\mathcal{O}_F(x)$ in a space-time diagram which consists in piling up the successive iterates $x, F(x), F^2(x) \dots$ (see Figure 2).

Traces In subspaces of $A^{\mathbb{M}}$, the continuity implies some concept of “locality”; we can study what happens to some portion of the configuration. This notion is called *trace* and is depicted in Figure 2. It can be formally defined as follows:

Definition 4. *The trace application of some TDDS $(\Sigma \subset A^{\mathbb{M}}, F)$ in cells $\llbracket i, k \rrbracket$, where $i, k \in \mathbb{M}$ and $i < k$, is:*

$$\begin{aligned} T_F^{\llbracket i, k \rrbracket} : \Sigma &\rightarrow \mathcal{L}_{k-i}(\Sigma)^{\mathbb{N}} \\ x &\mapsto (F^t(x)_{\llbracket i, k \rrbracket})_{t \in \mathbb{N}} . \end{aligned}$$

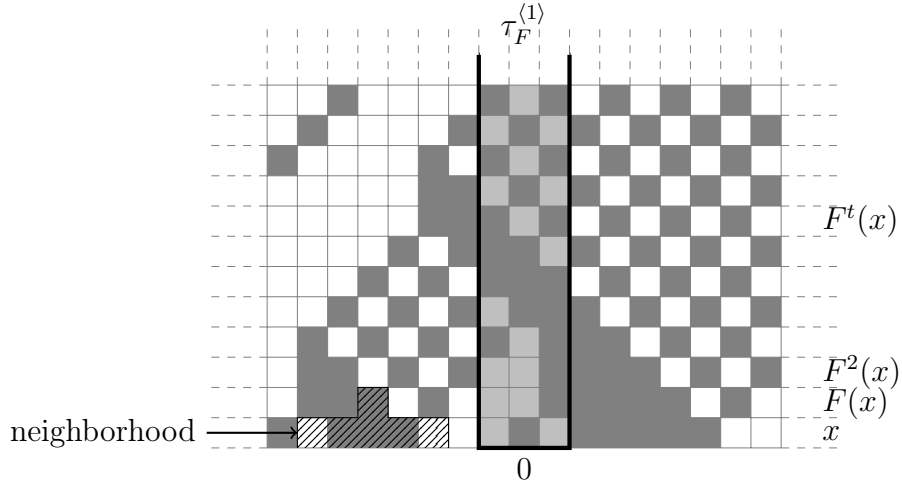


Figure 2. Space-time diagram and trace over segment $\llbracket -1, 1 \rrbracket$

The image $\tau_F^{\llbracket i, k \rrbracket} = T_F^{\llbracket i, k \rrbracket}(\Sigma)$ is a onesided subshift over alphabet $\mathcal{L}_{k-i}(\Sigma)$ and $T_F^{\llbracket i, k \rrbracket}$ is a factor map of (Σ, F) onto $(\tau_F^{\llbracket i, k \rrbracket}, \sigma)$. In TDDS, finer and finer traces can be used to approach the global system, like observations made with some error represented by the partition. We will note $\tau_F = \tau_F^0$ the central trace.

The traces of some PCA (Σ, F) have a very specific property: for any $i, k, h \in \mathbb{M}$, $\tau_F^{\llbracket i, k \rrbracket} = \tau_F^{\llbracket i+h, k+h \rrbracket}$ thanks to invariance by shift. In particular, we get the following basic property.

Proposition 1. *If (Σ, F) is a PCA of diameter $d \in \mathbb{N}$ and anchor $m \in \llbracket 0, d \rrbracket$ on some $(d-1)$ -SFT, $q \in \mathbb{N} \sqcup \{\infty\}$, $i \in \mathbb{M}$ and $x, y \in \Sigma$ two configurations such that $T_F^{\llbracket i, i+d-1 \rrbracket}(x)_{\llbracket 0, q \rrbracket} = T_F^{\llbracket i, i+d-1 \rrbracket}(y)_{\llbracket 0, q \rrbracket}$. Then for any generation $t \in \llbracket 0, q \rrbracket$, $F^t(x \oplus_i y) = F^t(x) \oplus_i F^t(y)$.*

Note that the conditions over the anchor and the order of the SFT are not so restrictive, since we can always enlarge the diameter.

Proof. We can see by recurrence on $t < q$ that the neighborhood $F^t(x \oplus_i y)_{\llbracket k-m, k-m+d \rrbracket}$ of each cell $k \in \mathbb{M}$ corresponds to the neighborhood $F^t(x)_{\llbracket k-m, k-m+d \rrbracket}$ if $k < m$, $F^t(y)_{\llbracket k-m, k-m+d \rrbracket}$ otherwise; therefore the application of the local rule remains unchanged. \square

Nilpotency, preperiodicity We are first interested in the DDS where the dynamics of every point are ultimately very simple (either stable or periodic). Let (X, F) a DDS, $z \in X$ a point. A point $x \in X$ is said *z-nilpotent* if there exists a generation $q \in \mathbb{N}$ such that for any $t \geq q$, $F^t(x) = z$. It is said (p, q) -*preperiodic* if $p, q \in \mathbb{N}$ such that $F^{p+q}(x) = F^q(x)$. The system is said to be *weakly z-nilpotent* (resp. *weakly preperiodic*) if all of its points are *z-nilpotent* (resp. *preperiodic*). These definitions allow stronger versions when conditions are uniformized on every points, as follows.

Definition 5. A DDS (X, F) is said z -nilpotent, for $z \in X$, if there exists a generation $q \in \mathbb{N}$ such that for any $t \geq q$, $F^t(X) = \{z\}$. It is said p -periodic (resp. (p, q) -preperiodic) if $F^p = \text{id}$ (resp. $F^{p+q} = F^q$).

The value q is called the *preperiod* and p the *ultimate period*. We will sometimes speak of periodic, preperiodic or p -preperiodic DDS.

The finite DDS are exactly the preperiodic subshifts – up to conjugacy.

If $(\Sigma \subset A^{\mathbb{M}}, F)$ is a PCA, $0 \in A$ and $t \in \mathbb{N}$ a generation such that for any configuration $x \in \Sigma$, $F^t(x)_0 = 0$, then it can be seen (thanks to shift-invariance) that F is nilpotent. Conversely, if $(A^{\mathbb{M}}, F)$ is a CA which is not 0-nilpotent for some $0 \in A$, then for any $t \in \mathbb{N}$, $F^{-t}([A \setminus \{0\}])$ is nonempty and open; in particular, it contains some 0-finite configuration. Moreover, looking at the dynamics of the finite subsystem of uniform configurations, we can see that for any CA $(A^{\mathbb{M}}, F)$, there is a generation $p \in \llbracket 0, |A| \rrbracket$ and a state $0 \in A$ which is quiescent for the CA F^p , in such a way that the set of 0-finite configurations is F^p -invariant. Summing up the two previous points, we get that for any generation $t \in \mathbb{N}$, $F^t(A^{\mathbb{M}})$ contains some 0-finite nonuniform configuration $z = {}^\infty 0[u]0^\infty$, with $u \in A^*$ and $z_0 \neq 0$. Last remark but not least, both nilpotency and preperiodicity are preserved under simulation.

We can actually prove that the classes of weakly nilpotent or periodic CA collapse to their strong counterpart.

Proposition 2. Any weakly preperiodic (resp. weakly nilpotent) PCA (Σ, F) over some transitive subshift Σ is preperiodic (resp. nilpotent).

Proof. Our hypothesis consists in decomposing the compact set Σ of nonempty interior into the union of subshifts $\bigcup_{q \in \mathbb{N}} \bigcup_{p \in \mathbb{N}_+} F^{-q}(\{x \in A^{\mathbb{M}} \mid F^p(x) = x\})$ (resp. $\bigcup_{q \in \mathbb{N}} F^{-q}(\{z\})$). By Baire's theorem, one of them has nonempty interior, and thus contains a configuration x which is transitive for σ . As a subsystem, it shall contain also $\overline{\mathcal{O}_\sigma(x)} = \Sigma$. \square

Transitivity, recurrence, nonwanderingness Up to now, we have looked at properties regarding each orbit independently. Let us now take benefit of the topology to study how distinct orbits from a single open set behave. For a DDS (X, F) , a point $x \in X$ is said *transitive* if its positive orbit is dense: $\overline{\mathcal{O}_F^+(x)} = X$. It is said *recurrent* if for any neighborhood U of x , there is some generation $t > 0$ such that $F^t(x) \in U$. It is *nonwandering* if for any neighborhoods U, V of x , there is some point $y \in U$ and some generation $t > 0$ such that $F^t(y) \in V$. Those definitions can be extended to dynamical systems. A DDS (X, F) is *transitive* if it admits a residual subset of transitive points and *nonwandering* if all of its points are nonwandering.

By compactness, we can have the following equivalent characterization: a DDS (X, F) is *transitive* if and only if for any nonempty open sets $U, V \subset X$, there is some point $x \in U$ and some generation $t > 0$ such that $F^t(x) \in V$. It is *nonwandering* if any nonempty open set $U \subset X$ contains some point $x \in U$ and some generation $t > 0$ such

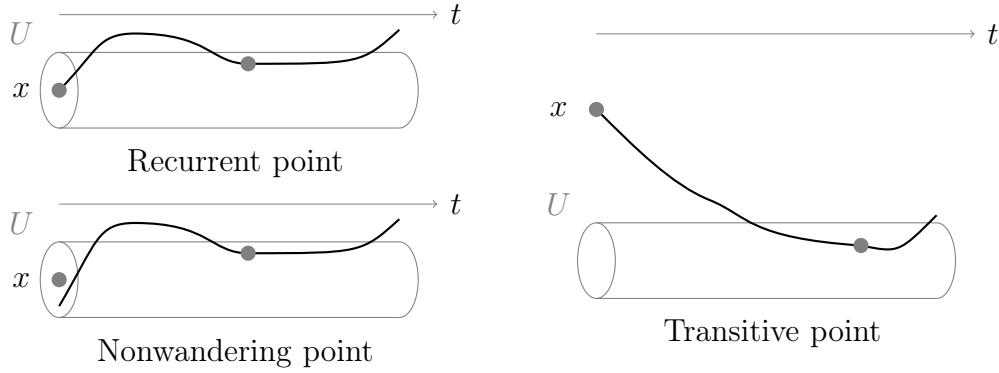


Figure 3. Recurrence and nonwanderingness

that $F^t(x) \in U$. This is equivalent to having only nonwandering points and to having a residual set of recurrent points, by the following proposition (see [9]).

Proposition 3. *Any nonwandering DDS has residual set of recurrent points.*

Proof. If (X, F) is a DDS, the set of its recurrent points can be written $\mathcal{R} = \bigcap_{n \in \mathbb{N}} \mathcal{R}_n$, where $\mathcal{R}_n = \{x \in X \mid \exists t > 0, d(F^t(x), x) < 1/n\}$ for all $n \in \mathbb{N}$. Note that \mathcal{R}_n is open. Moreover, if $x \in X$ is nonwandering and $\varepsilon = 1/2n$, then by definition there exists some neighbor point $y \in \mathcal{B}_\varepsilon(x)$ and some generation $t > 0$ such that $F^t(y) \in \mathcal{B}_\varepsilon(x)$; in particular $y \in \mathcal{R}_n$. We have proved that each \mathcal{R}_n is dense, which gives the result by Baire's theorem. \square

Equicontinuity, sensitivity Now, let us take some alternative observation method and study how large changes can appear when introducing a small change in the configuration.

Let (X, F) a DDS, $\varepsilon \in \mathbb{R}_+ \setminus \{0\}$. A point $x \in X$ is said ε -unstable if for any radius $\delta > 0$, there is a point $y \in \mathcal{B}_\delta(x)$ and a generation $t \in \mathbb{N}$ for which $d(F^t(x), F^t(y)) > \varepsilon$. Otherwise the point is said ε -stable (see Figure 4). A point which is ε -stable for any $\varepsilon > 0$ is said *equicontinuous*.

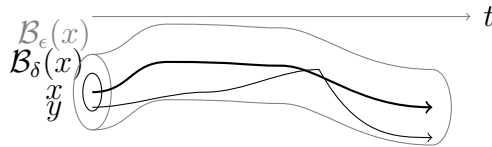


Figure 4. An ε -stable point.

Definition 6. *A DDS F is said ε -sensitive if all of its points are ε -unstable, with $\varepsilon > 0$. It is said almost equicontinuous if its set of equicontinuous points is a residual. It is equicontinuous if for any radius $\varepsilon > 0$, there exists a radius $\delta > 0$ such that for all points $x, y \in X$ with $d(x, y) < \delta$ and all generation $t \in \mathbb{N}$ we have $d(F^t(x), F^t(y)) < \varepsilon$.*

Due to the compactness of the underlying space, it is possible to invert the two quantifiers in the definition of equicontinuity and to achieve the following characterization: a DDS F is equicontinuous if and only if all of its points are.

A first example of equicontinuous systems is the preperiodic ones: their behavior only depends on the beginning of their orbit.

Proposition 4. *Any preperiodic DDS is equicontinuous.*

Proof. Let (X, F) be a (p, q) -preperiodic DDS with $q \in \mathbb{N}$, $p \in \mathbb{N}_+$ and $\varepsilon > 0$. Then each iterate F^t , for $t \in \mathbb{N}$, is uniformly continuous, *i.e.* there exists $\delta_t > 0$ such that for all points $x, y \in X$ with $d(x, y) < \delta_t$, we have $d(F^t(x), F^t(y)) < \varepsilon$. Since $F^t = F^{q+(t-q) \bmod p}$, we can define $\delta = \min_{0 \leq j < p+q} \delta_j$, in such a way that for all generation $t \in \mathbb{N}$ and all points $x, y \in X$ with $d(x, y) < \delta$, we have $d(F^t(x), F^t(y)) < \varepsilon$. \square

In the specific case of a TDDS (Σ, F) , the definition of an equicontinuous point $x \in \Sigma$ can be reformalized in terms of the trace as follows: for any $k \in \mathbb{N}$, there exists $l \in \mathbb{N}$ such that $T_F^{(k)}([x_{\langle l \rangle}])$ is a singleton. We can further characterize the notion of equicontinuity as follows.

Proposition 5. *A TDDS (Σ, F) is equicontinuous if and only if all of its traces are finite.*

Proof. Let F be an equicontinuous TDDS and $k \in \mathbb{N}$. There exists a radius $l \in \mathbb{N}$ such that for any $u \in A^{\langle l \rangle}$, $T_F^{(k)}([u])$ is a singleton. Consequently, $\tau_F^{(k)} = \bigcup_{u \in A^{\langle l \rangle}} T_F^{(k)}([u]) \leq |A^{\langle l \rangle}|$.

Conversely, if $\tau_F^{(k)}$ is finite, then it is (p, q) -preperiodic, for some $p \in \mathbb{N}_+$ and $q \in \mathbb{N}$. Any point $x \in \Sigma$ is ε -stable, since any point y of the neighborhood $\bigcap_{t < p+q} F^{-t}(\mathcal{B}_\varepsilon(F^t(x)))$ satisfies $\forall t \in \mathbb{N}, d(F^t(x), F^t(y)) < \varepsilon$. \square

In the very particular case of PCA, homogeneity allows to note that if all the cells have preperiodic traces with the same period and preperiod, then the whole configuration is preperiodic. Thus, we can state that a PCA F is preperiodic if and only if each of its traces $\tau_F^{(k)}$, with $k \in \mathbb{N}$, is finite. The trace of width 1 being the projection of all other traces, the period and preperiod can be uniformized, which leads to a simple generalization of a classical result over CA or PCA on very particular subshifts [10, 11].

Corollary 6. *Any PCA is equicontinuous if and only if it is preperiodic.*

Concerning sensitivity, it is not transmitted to any trace, but it is to sufficiently fine traces as shown by the following proposition.

Proposition 7. *Let (Σ, F) be an ε -sensitive TDDS with $\varepsilon \geq 2^{-k}$. Then $\tau_F^{(k)}$ is a sensitive subshift.*

Proof. Let $x \in \Sigma$ and $\delta > 0$. By continuity of the trace application, there exists $\delta' > 0$ such that for any configuration $y \in \mathcal{B}_{\delta'}(x)$, we have $d(T_F^{(k)}(x), T_F^{(k)}(y)) < \delta$. The sensitivity of F gives a configuration $y \in \mathcal{B}_{\delta'}(x)$ and a generation $t \in \mathbb{N}$ such

that $d(F^t(x), F^t(y)) > \varepsilon$, i.e. $F^t(x)_{\langle k \rangle} \neq F^t(y)_{\langle k \rangle}$. As a result, $T_F^{(k)}(x)_t \neq T_F^{(k)}(y)_t$, i.e. $d(\sigma^t T_F^{(k)}(x), \sigma^t T_F^{(k)}(y)) = 1$, with $d(T_F^{(k)}(x), T_F^{(k)}(y)) < \delta$. \square

In the space $A^{\mathbb{M}}$, stability can be linked with blocking words, defined as follows: a word $w \in A^*$ is k -blocking for the TDDS (Σ, F) if there exists $i \in \mathbb{N}$ such that $\forall x, y \in [w]_{-i}, \forall j \in \mathbb{N}, F^j(x)_{[0, k]} = F^j(y)_{[0, k]}$.

The reader can easily note that a word is k -blocking if one of its patterns is, and that any k -blocking word is i -blocking for all $i \leq k$. Moreover, if (Σ, F) is a TDDS and $k \in \mathbb{N}$, then a configuration $x \in \Sigma$ is 2^{-k} -stable if and only if $x_{\langle l \rangle}$ is k -blocking for some $l \in \mathbb{N}$, which brings the following remark.

Remark 1.

- A TDDS is 2^{-k} -sensitive if and only if it does not admit any k -blocking word.
- A configuration is equicontinuous if and only if it admits k -blocking central patterns for any $k \in \mathbb{N}$.

But blocking words are especially interesting regarding CA, since a particular width is enough to block all widths. Intuitively, these blocking words will disconnect the underlying space into two different components, preventing future information transfers between them.

If (Σ, F) is a PCA of radius r , w a r -blocking word, $i \in \mathbb{N}$ as in the definition, and $x \in [w]_{-i}$, then for any configuration $y \in \Sigma$ with $y_{[-i, \infty[} = x_{[-i, \infty[}$ (resp. $y_{]-\infty, |w|-i]} = x_{]-\infty, |w|-i]}$) and any generation $t \in \mathbb{N}$, we have $F^t(y)_{[0, \infty[} = F^t(x)_{[0, \infty[}$ (resp. $F^t(y)_{]-\infty, r]} = F^t(x)_{]-\infty, r]}$).

For instance, in the Min CA, the word 0 is 1-blocking, since $\forall x \in [0], \forall t \in \mathbb{N}, F^t(x)_0 = 0$; its radius being 1, any space-time diagram containing 0 can be separated into two parts evolving independently.

Remark 2. Let (Σ, F) be a PCA of radius $r \in \mathbb{N}$, $i, j \in \mathbb{M}$, $k, l \geq r$, and u, v two words which are respectively k -blocking and l -blocking, i and i' the corresponding indices in the words (from the definition). If the concatenation uv is in the language $\mathcal{L}(\Sigma)$, then it is $|u| - i + j + l$ -blocking.

This last fact implies the following proposition.

Proposition 8 (Kůrka [10]). Let (Σ, F) a PCA of radius r . Then F is equicontinuous if and only if there exists $k \in \mathbb{N}$ such that all the words of A^k are r -blocking.

Another consequence of Remark 2 is that we can insert any word between two concatenated words and obtain arbitrarily wide blocking words in any cylinder. We obtain the following theorem, equivalent to a result in [10].

Theorem 2. Let (Σ, F) a PCA of radius r on some transitive subshift. The following statements are equivalent:

- (i) F is almost equicontinuous;
- (ii) F is not 2^r -sensitive;

(iii) F admits some r -blocking word.

Proof.

- Assume that $u \in A^*$ is a k -blocking word for F , with $k \geq r$, and let us show that the set of equicontinuous configurations is a residual. By transitivity of Σ , the open set $U_l = \bigcup_{j>l} [u]_j$ is dense. Thanks to the Remarks 2 and 1, the configurations of the intersection $\bigcap_{l \in \mathbb{N}} U_l$ are equicontinuous.
- The other implications directly come from the definitions. \square

2. Limit set

The previous section has presented many different possible ways to study dynamics of DDS and some first results about them. However, these notions can be very sensitive to the transient time (if we modify the initial evolution during a short time). To overcome this problem and characterize the core behavior of the DDS, one idea is to consider only the points that can appear arbitrarily late inside the DDS. This corresponds to the limit set. Formally, if (X, F) is a DDS and $X' \subset X$, we note $\Omega_F(X') = \bigcap_{j \in \mathbb{N}} \overline{\mathcal{O}_F(F^j(X'))}$. In case of an invariant set (such as X), the definition gets simpler.

Definition 7. Let (X, F) a DDS. The limit set of a DDS (X, F) is $\Omega_F = \Omega_F(X) = \bigcap_{j \in \mathbb{N}} F^j(X)$.

The limit set of the Min CA is the set of the configurations where all the 1s are connected:

$$\Omega_{\text{Min}} = \{x \in 2^{\mathbb{Z}} \mid \forall i \in \mathbb{Z}, x_i = 0 \Rightarrow \forall j < i, x_j = 0 \text{ or } \forall j > i, x_j = 0\}.$$

2.1. Limit set of DDS

One first easy remark is that the limit set is always a closed nonempty set, as a decreasing intersection of nonempty closed subsets.

The limit set corresponds to the largest surjective subsystem; in particular $\Omega_F = X$ if and only if F is onto. Let us now define an *attractor* as a set that attracts neighboring points or, formally, a nonempty closed F -invariant subset Y of X such that for any $\varepsilon > 0$, there exists $\delta > 0$ such that for any point $x \in X$ with $d(x, Y) < \delta$, we have $\lim_{j \rightarrow \infty} d(F^j(x), Y) = 0$ and for any generation $j \in \mathbb{N}$, $d(F^j(x), Y) < \varepsilon$. The limit set is then the maximal attractor. Using this characterization and the compactness of the underlying space, it can be shown that any neighborhood is reached in a finite time, *i.e.* $\max_{x \in X} d(F^j(x), \Omega_F) \rightarrow_{j \rightarrow \infty} 0$. In the case where the limit set is reached in a finite uniform time, *i.e.* there exists a generation $j \in \mathbb{N}$ such that $F^j(X) = \Omega_F$, we say that the DDS (X, F) is *stable*.

If (X_1, F) and (X_2, F) are two subsystems of (X, F) such that $X_1 \cup X_2 = X$, then $\Omega_F = \Omega_F(X_1) \cup \Omega_F(X_2)$.

We can build, using the limit set, similar notions to those we already introduced.

Definition 8. Let (X, F) be a DDS. It is said z -limit-nilpotent for some $z \in X$, if Ω_F is the singleton $\{z\}$. It is said p -limit-periodic, with $p \in \mathbb{N}_+$, if $F|_{\Omega}$ is p -periodic (i.e., $F|_{\Omega}^p = Id$).

A first remark is that any z -nilpotent DDS is z -limit nilpotent, and similarly preperiodic DDS are limit-periodic, but the converse is not always true. Nevertheless these new notions represent some highly stable behavior, as seen in the following proposition.

Proposition 9. Any limit-nilpotent DDS is equicontinuous.

Proof. Let (X, F) be a z -limit-nilpotent DDS for some $z \in X$, and $\varepsilon > 0$. There exists a generation $J \in \mathbb{N}$ such that $\forall j \geq J, \max_{x \in X} d(F^j(x), z) < \frac{\varepsilon}{2}$. Hence, for any point $x \in X$ and any point y of the open set $\bigcap_{0 \leq j < J} F^{-j}(\mathcal{B}_\varepsilon(F^j(x)))$, we have by construction $d(F^j(x), F^j(y)) < \varepsilon$ for $j < J$ and $d(F^j(x), F^j(y)) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$. It results that x is ε -stable. \square

Moreover, we can remark that a system is p -limit-periodic if and only if its limit set is the set of its p -periodic points.

The limit set of a subsystem is included in that of the whole system. Hence any subsystem of a limit-nilpotent (resp. limit-periodic) is limit-nilpotent (resp. limit-periodic). Moreover, the limit set is preserved under iteration and factor map. Thus, we can see that if Φ is a simulation by a DDS (X, F) of another (Y, G) then $\Omega_G(Y) \subseteq \Phi(\Omega_F(X))$, and we have the following.

Proposition 10. Let $\Phi : X \rightarrow Y$ be a complete simulation by a DDS (X, F) of another (Y, G) . Then $\Phi(\Omega_F(X)) = \Omega_G(Y)$.

Proof. Let Φ be a factor map. For any $j \in \mathbb{N}$, $\Phi F^j(X) = G^j(Y)$, hence $\Phi(\bigcap_{j \in \mathbb{N}} F^j(X)) = \bigcap_{j \in \mathbb{N}} G^j(Y)$ since it is a decreasing intersection; hence $\Phi(\Omega_F) = \Omega_G$. Moreover, decreasingness of the sequence clearly gives $\Omega_{F^k} = \Omega_F$ for any $k \in \mathbb{N}_+$. \square

2.2. Limit set of subshifts

Here, we will study what happens when we are in the case of TDDS or, in particular, of onesided subshifts (obviously, the limit system of a twosided subshift is the whole subshift). For a onesided subshift Σ , we will more conveniently note $\Omega_\Sigma = \Omega_\sigma(\Sigma)$.

For the label system of a graph, the limit set can be read by removing the inaccessible vertices (until none remain). More formally, the limit set of the label system of a given graph is the subgraph composed of vertices that are accessible by an infinite path. In this case, we have the property that sofic subshifts are stable. Actually, as soon as the limit set is an SFT, the next proposition and corollary show that it is reached in finite time.

Proposition 11. Any subshift having a limit set of finite type is stable.

Proof. Let Σ be a subshift such that Ω_Σ is an SFT of order $k \in \mathbb{N}$. $[\mathcal{L}_k(\Omega_\Sigma)]$ is a neighborhood of Ω_Σ , so it is reached in finite time (from previous remarks): there exists a generation $t \in \mathbb{N}$ for which $\sigma^t(\Sigma)$ is included in $[\mathcal{L}_k(\Omega_\Sigma)]$. Being a subshift, $\sigma^t(\Sigma)$ must also be included in $\bigcap_{j \in \mathbb{N}} \sigma^j([\mathcal{L}_k(\Omega_\Sigma)])$, which is exactly Ω_Σ since it is an SFT of order k . \square

Corollary 12. *A subshift is finite (resp. of finite type) if and only if its limit set is.*

Proof. Let $\Sigma \subset A^\mathbb{M}$ be a subshift such that Ω_Σ is an SFT of order $k \in \mathbb{N}_+$. By proposition 11, there exists a generation $j \in \mathbb{N}$ such that $\sigma^j(\Sigma) = \Omega_\Sigma$. It is then immediate that Σ is a $(k + j)$ -SFT and that $|\Sigma| \leq |A^j| |\Omega_\Sigma|$. The converse is immediate by the previous remark on the limit set of a label system. \square

In other words, the limit-periodic (resp. limit-nilpotent) subshifts are exactly the preperiodic (resp. nilpotent) subshifts. The argument of the previous proof cannot be adapted to sofic subshifts, as shown by the following counter-example: the subshift $\{0^k 1^l 0^\infty \mid k \leq l\} + 0^* 1^\infty$ is not sofic, even though its limit set $1^* 0^\infty + 0^* 1^\infty$ is sofic.

2.3. Limit set of cellular automata

In the case of (partial) cellular automata or TDDS, the particular structure allows stronger results. Nevertheless, it is not completely understood, as suggests the attempt to characterize the possible limit sets of CA in [12], or more generally the possible subshift attractors in [13], also linked to [14]. Generally, it is known that the limit sets of CA can be rather complex [15, 16].

First note that the limit set of a PCA (Σ, F) is a subshift, as an intersection of subshifts. Moreover, its language is the limit $\mathcal{L}(\Omega_F) = \bigcap_{j \in \mathbb{N}} \mathcal{L}(F^j(\Sigma))$ of the languages of the successive image subshifts. In the case of TDDS, we can generalize Proposition 9 to obtain a strong condition of stability via the limit set.

Proposition 13. *Any limit-periodic TDDS is equicontinuous.*

Proof. Let (Σ, F) be a limit-periodic TDDS. Then Proposition 10 gives that its traces are all limit-periodic too, and Corollary 12 that they are preperiodic. Proposition 5 allows then to conclude that F is equicontinuous. \square

As far as nilpotency is concerned, it is obvious that a TDDS is nilpotent if and only if all of its traces are nilpotent. For a PCA F , as all of them share the same projection of width 1, the characterization is simpler: F is nilpotent if and only if the central trace τ_F is nilpotent. The case of period $p = 1$ gives us a generalization of a well-known characterization of CA nilpotency [5].

Proposition 14. *Any PCA is nilpotent if and only if it is limit-nilpotent.*

In the case of a full CA, we can prove some restriction on the limit set showing that the nilpotent behavior can be “isolated” from other behaviors: if a CA is not nilpotent, its limit set will contains numerous configurations.

Proposition 15. *Let $(A^{\mathbb{M}}, F)$ a non 0-nilpotent CA, with $0 \in A$ and ${}^\infty 0^\infty \in \Omega_F$. Then Ω_F contains, for any $k \in \mathbb{M}$, a semifinite configuration $z \neq {}^\infty 0^\infty$ such that $z_i = 0$ for all cell $i < k$.*

Proof. We can consider without loss of generality that the CA is twosided. By a previous remark, for any generation $j \in \mathbb{N}$, $F^j(A^{\mathbb{Z}})$ contains some 0-finite nonuniform configuration $z \in {}^\infty 0[u]0^\infty$, with $u \in A^+ \setminus 0^+$. Composing with a shift, we obtain $F^j(A^{\mathbb{M}}) \cap {}^\infty 0[0^C]_k \neq \emptyset$ and compactness gives $\Omega_F \cap {}^\infty 0[0^C]_k \neq \emptyset$. \square

As a consequence, we have another characterization of nilpotency.

Corollary 16. *A CA is nilpotent if and only if it admits some isolated uniform configuration.*

Proof. Let $(A^{\mathbb{M}}, F)$ a non-nilpotent CA, $0 \in A$, $k \in \mathbb{N}$; Proposition 15 gives some nonuniform configuration in $\mathcal{B}_{2-k}(x) \cap \Omega_F$. The converse is obvious. \square

The previous result allows us to obtain some well-known fact on the cardinality of the limit set of a cellular automaton.

Proposition 17 (Čulík, Pachl & Yu [5]). *The limit set of any CA is either a singleton or infinite.*

An infinite limit set can be countable as the Min CA, or uncountable, as for surjective CA, in which case, being a subshift, it has a continuous cardinality. The dichotomy of the previous proposition is no more true for PCA, for instance on finite subshifts.

3. Asymptotic set

If the limit set characterizes the set of points that can appear arbitrarily late during the evolution of the dynamical systems, it may actually contain points which look transient. This is the case of configurations of the form ${}^\infty 01 \dots 10^\infty$ for the Min CA: we know they will disappear soon. To better emphasize the asymptotic behavior, we study here the set containing all the points for which there exists an evolution of the dynamical system going an infinite number of times close to this point.

Definition 9. *Let (X, F) be a DDS. The asymptotic set of a set $X' \subset X$ is the set $\omega_F(X') = \bigcup_{x \in X'} \Omega_F(\{x\})$ of adhering values of orbits. We note $\omega_F = \omega_F(X)$.*

This set was called *ultimate set* in [17, 7], or *accessible set* in [18]. For instance, the asymptotic set of the Min CA is $\{{}^\infty 0^\infty, {}^\infty 1^\infty\}$, and is strictly included in its limit set.

3.1. Asymptotic set of DDS

Like the limit set, the asymptotic set can be expressed by a metric property: it is the smallest subset $Y \subset X$ such that for any $x \in X$, $d(F^j(x), Y) \rightarrow_{j \rightarrow \infty} 0$. In other words, for any neighborhood U of ω_F and any point $x \in X$, there exists a generation $J \in \mathbb{N}$ such that $\forall j \geq J, F^j(x) \in U$.

We can immediately see that if $X' \neq \emptyset$, then $\omega_F(X')$ is nonempty and F -invariant, but need not be closed (as opposed to the limit set). The asymptotic set is also always a subset of the limit set: $\omega_F(X') \subset \Omega_F(X')$. One important problem is to understand the dynamics of the orbits which are in the difference of the two sets. First note that all the periodic points are contained in the asymptotic set. The following propositions go further.

Proposition 18. *The asymptotic set of a DDS (X, F) contains all of its transitive subsystems.*

Proof. Consider a transitive subsystem $(Y \subset X, F)$. Then there is a point $y \in Y$ which is transitive for this subsystem, i.e. any point of Y is an adhering value of $\mathcal{O}_F(y)$. \square

Example 3 will show that the inclusion can be strict.

It is known that the set of *uniformly recurrent* points is the union of the *minimal* subsystems (see for instance [19] for definitions). Similarly, we can prove the following proposition.

Proposition 19. *For any recurrent point x , the subsystem $\overline{\mathcal{O}_F(x)}$ is transitive.*

Proof. In general, the closure $\overline{\mathcal{O}_F(x)}$ of the orbit is the union of the closure $\overline{\mathcal{O}_F^+(x)}$ of the positive orbit and of the singleton x . By the property of recurrence, $x \in \overline{\mathcal{O}_F^+(x)}$. Hence x has a dense positive orbit in this subsystem. \square

Nevertheless, the set of transitive subsystems also includes other points: see for instance the case of the full shift, which is transitive, but admits some non-recurrent points. The two last propositions give that the asymptotic set contains the set of recurrent points; actually it can easily be seen that they are exactly the points which are an adhering value of their own orbit. On the other hand, we can show that it is a subset of the set of nonwandering points.

Proposition 20. *Any point of the asymptotic set of a DDS is nonwandering.*

Proof. Let (X, F) a DDS, $\varepsilon > 0$ and $x \in \omega_F$, i.e. there exists a point $y \in X$ whose orbit admits x as adhering value; in particular, it goes an infinite number of times in the ball $\mathcal{B}_\varepsilon(x)$. Therefore, there exist some point $y' = F^J(y) \in \mathcal{B}_\varepsilon(x)$ and some generation $j \in \mathbb{N}_+$ such that $F^j(y') = F^{J+j}(y) \in \mathcal{B}_\varepsilon(x)$. \square

The main interest in these inclusions is that they are “not far” from each other: from the remark that the set of nonwandering points is closed and from Proposition 3,

we deduce the following characterization: a DDS (X, F) is nonwandering if and only if its asymptotic set ω_F is a residual subset of X .

The long-term behavior of the orbits of a system tends to look more and more like the behavior on the asymptotic set, it is therefore relevant to study the case when asymptotic points have a simple evolution, as we have done for the limit set.

Definition 10. A DDS (X, F) is asymptotically z -nilpotent if all of its orbits converge towards the same limit $z \in X$, i.e. $\omega_F = \{z\}$. It is asymptotically p -periodic, with $p \in \mathbb{N}_+$, if the restricted map $F|_{\omega_F}$ is p -periodic.

With these definitions, if F is an asymptotically z -nilpotent DDS, then z is a fix point of F , since ω_F is F -invariant; in particular, F is asymptotically 1-periodic. Moreover, for any $\varepsilon > 0$, there exists a generation $J \in \mathbb{N}$ such that for any point $x \in X$, $\exists j < J, d(F^j(x), z) < \varepsilon$.

It is possible to link these behaviors with the previously-defined ones. The first easy point is that weakly nilpotent DDS are asymptotically nilpotent. However, the converse is not true. A simple counter-example is the division by 2 on interval $[0, 1]$. Nevertheless, asymptotically nilpotent DDS cannot be too much unstable, as formalized by the following proposition.

Proposition 21. No asymptotically nilpotent DDS is sensitive.

Proof. Let (X, F) an asymptotically z -nilpotent DDS, with $z \in X$, and $\varepsilon > 0$. By definition, the space X , of nonempty interior, can be decomposed as a union $\bigcup_{J \in \mathbb{N}} \bigcap_{j > J} F^{-j}(\mathcal{B}_{\varepsilon/2}(z))$ of closed subsets. By Baire's theorem, there exists a generation $J \in \mathbb{N}$ such that the closed subset $\bigcap_{j > J} F^{-j}(\mathcal{B}_{\varepsilon/2}(z))$ contains an open subset U of nonempty interior. Let $x \in U$. The finite intersection $U \cap \bigcap_{j \leq J} F^{-j}(\mathcal{B}_{\varepsilon}(F^j(x)))$ is then open and contains x ; consequently, it contains an open ball $\mathcal{B}_{\delta}(x)$, with $\delta > 0$. For any point $y \in \mathcal{B}_{\delta}(x)$ and any generation $j \leq J$, we have by construction $d(F^j(x), F^j(y)) \leq \varepsilon$; for any generation $j > J$, we have the triangular inequality $d(F^j(x), F^j(y)) \leq d(F^j(x), z) + d(z, F^j(y)) \leq \varepsilon$. As a result, the point x is ε -stable. \square

Let us look at how the asymptotic set can be related to the notions of simulation.

Proposition 22. Let Φ a complete simulation by a DDS (X, F) of another (Y, G) . Then $\Phi(\omega_F(X)) = \omega_G(Y)$.

Proof. Suppose that Φ is a factor map. Let $x \in \omega_G$, i.e. x is the limit of a subsequence $(G^{k_j}\Phi(y))_{j \in \mathbb{N}}$ where $(k_j)_{j \in \mathbb{N}}$ is an increasing sequence of integers. Then $(F^{k_j}(y))_{j \in \mathbb{N}}$ admits a adhering value z , whose image is $\Phi(z) = x$. Hence $x \in \Phi(\omega_F)$. The converse is immediate.

It is now sufficient to show that any DDS F^k has the same asymptotic set than F . First, the decreasingness of the sequence gives $\omega_{F^k} \subset \omega_F$. Then, let $x \in \omega_F$, i.e. x is the limit of some subsequence $(F^{k_j}(y))_{j \in \mathbb{N}}$, where $(k_j)_{j \in \mathbb{N}}$ is an increasing sequence of integers. By the pigeon-hole principle, there exists some integer $r < k$ such that

$J = \{j \in \mathbb{N} \mid k_j \bmod k = r\}$ is infinite. We can see that $(F^{k_j-r}(y))_{j \in J}$ is a subsequence of the orbit of $F^r(y)$ by F^k that admits x as an adhering value. Hence $x \in \omega_{F^k}$. \square

Moreover, the asymptotic set of some subsystem is contained in the asymptotic set of the global system. We even have that, if (X_1, F) and (X_2, F) are two subsystems of (X, F) such that $X_1 \cup X_2 = X$, then $\omega_F = \omega_F(X_1) \cup \omega_F(X_2)$. In particular, like nilpotency and preperiodicity, asymptotic nilpotency and asymptotic periodicity are transmitted by any simulation.

3.2. Asymptotic set of subshifts

Let us study how the asymptotic set is constrained in the particular case of subshifts. To lighten the reading, we will note $\omega_\Sigma = \omega_\sigma(\Sigma)$.

We can see that the asymptotic set of sofic subshifts can be seen from the graph of their limit sets by removing all the links between strongly connected components. That result can also be restated as follows: the asymptotic set of a sofic subshift is the disjoint union of its maximal transitive subsystems.

In particular, in the onesided case, the asymptotic set is reached by each orbit: if Σ is a onesided sofic subshift and $z \in \Sigma$, then there exists a generation $j \in \mathbb{N}$ such that $\sigma^j(z) \in \omega_\Sigma$.

Using regularity of sofic subshifts, it is possible to characterize the notion of asymptotic periodicity: a sofic subshift is asymptotically periodic if and only if it is the label system of some graph in which all the strongly connected components are cycles. In the case of onesided sofic subshifts, the reachability of the asymptotic set from any orbit shows that asymptotic periodicity is equivalent to weak preperiodicity. This can be generalized as follows.

Proposition 23. *Any onesided subshift is asymptotically periodic if and only if it is weakly preperiodic.*

Proof. Let $\Sigma \subset A^\mathbb{N}$ be an asymptotically periodic subshift of period $p \in \mathbb{N}_+$. The open subset $U = \{x \in \Sigma \mid x_0 = x_p\}$ is a neighborhood of ω_Σ . For any configuration $x \in \Sigma$, there is a generation $J \in \mathbb{N}$ such that for any $j \geq J$, $\sigma^j(x)_0 = \sigma^j(x)_p$, i.e. x is (J, p) -preperiodic. The converse is immediate. \square

We can use the previous proposition to get a generalization of Proposition 21 in that setting.

Corollary 24. *Any asymptotically periodic onesided subshift is almost equicontinuous.*

Proof. Let Σ be an asymptotically periodic onesided subshift of period $p \in \mathbb{N}_+$, and $\varepsilon > 0$. By Proposition 23, $\Sigma = \bigcup_{j \in \mathbb{N}} F^{-j}(\bigcap_{i \in \mathbb{N}} \{x \in \Sigma \mid x_i = x_{i+p}\})$. By Baire's theorem, there is some finite time $J \in \mathbb{N}$ and some nonempty open set $U \subset \bigcup_{j < J} F^{-j}(\bigcap_{i \in \mathbb{N}} \{x \in \Sigma \mid x_i = x_{i+p}\})$. If $x \in U$, then the intersection $V = U \cap \bigcap_{0 \leq j < J+p} \sigma^{-j}(\mathcal{B}_\varepsilon(\sigma^j(x)))$ is still open. For any $y \in V$ and any generation $j \in \mathbb{N}$,

we have $F^j(y) = F^{J+(j-J \bmod p)}(y)$ and $F^j(x) = F^{J+(j-J \bmod p)}(x)$; by construction, their distance is less than ε . Hence, x is ε -stable. We conclude recalling that any nonsensitive subshift is almost equicontinuous. \square

Actually, asymptotically periodic sofic subshifts are exactly those that have little simulation power, as suggested by the following proposition. We say that a subshift is *universal* if it can simulate all the other subshifts.

Proposition 25. *If Σ is a sofic subshift, the following statements are equivalent.*

- (i) Σ is not universal.
- (ii) Σ is countable.
- (iii) Σ is asymptotically periodic.
- (iv) Σ has no infinite transitive subsystem.
- (v) Σ is the label system of some graph with no non-cyclic strongly connected component.

Proof.

- ii \Rightarrow i: It is clear that a countable system cannot simulate an uncountable one (like a full shift on two letters).
- iii \Rightarrow ii: If Σ is asymptotically periodic, then we already noted that we can see it as the label system of a graph in which all the strongly connected components are cycles. Each configuration of Σ has a path that changes of strongly connected component only a finite number of times. The tuple of the indices of the cells that correspond to these changes of component and of the corresponding arc determines in a unique way the configuration. Σ is hence countable.
- iv \Rightarrow iii: We have seen that ω_Σ is the union of the maximal transitive subshifts of Σ . Hence, if $\sigma|_{\omega_\Sigma}$ is not periodic, then there is a transitive subsystem which is not periodic. On the other hand, it is known that transitive sofic subshifts are either cycles or infinite.
- v \Rightarrow iv: We know that any transitive subsystem of a sofic subshift is exactly the set of labels of a strongly connected components of some corresponding graph.
- i \Rightarrow v: If Σ is the label system of a graph (V, E) with some non-cyclic strongly connected component, then there exists three vertices v_0, v_1 and \tilde{v}_1 in this component and two distinct letters a and b such that $(v_0, v_1, a), (v_0, \tilde{v}_1, b) \in E$. By strong connectivity, there exists two paths $(v_i, w_i, u_i)_{0 \leq i \leq l}$ and $(\tilde{v}_i, \tilde{w}_i, \tilde{u}_i)_{0 \leq i \leq k}$ of respective lengths $l, k \in \mathbb{N}_+$ such that $v_0 = w_l = \tilde{v}_0 = \tilde{w}_k = v_0$. Let $u = (a\tilde{u})^{|b\tilde{v}|}$ and $v = (b\tilde{v})^{|a\tilde{u}|}$. We can see that $(u+v)^\infty$ is included in Σ . Moreover, it can be easily seen that $(\infty(u+v)^\infty, \sigma^{|u|})$ is conjugate to the full shift $(\mathbb{2}^{\mathbb{M}}, \sigma)$. The latter full shift is universal, since for any alphabet B , there is a trivial injection from B into $A^{\lceil \log |B| \rceil}$, which induces a conjugacy of any subshift over B onto some subsystem of the iterate $(\mathbb{2}^{\mathbb{M}}, \sigma^{\lceil \log |B| \rceil})$. \square

Let us now concentrate on the case of asymptotic nilpotency, which in particular implies the previous consequences of asymptotic preperiodicity. Note that if all configurations of a sofic subshift converge towards the same configuration, then this configuration is uniform and the subshift can be seen as the label system of a graph in which all the cycles share the same label. More formally, a sofic subshift is asymptotically nilpotent if and only if it contains a unique periodic configuration, which is then uniform. Using some of the previous results, it can equivalently be said that a sofic subshift is asymptotically nilpotent if and only if it is the label system of a graph in which each strongly connected component is a single arc, and all of them have the same label.

Using the particular case $p = 1$ in Proposition 23 gives us an equivalence between asymptotic nilpotency and weak nilpotency for onesided subshifts. This result is not true for general DDS.

The asymptotically nilpotent subshifts which are not nilpotent are actually rather complex. For instance, any asymptotically nilpotent subshift is an SFT if and only if it is nilpotent.

3.3. Asymptotic set of cellular automata

The intrinsic regularity of the model of CA allows more precise characterizations of asymptotic behaviors. We are going to present some of them, but maybe more are to be expected. The first easy remark is that the asymptotic set of a CA is shift-invariant. However, unlike the limit set, it need not be a subshift (see below) and can be arbitrarily complex (see for example [6]).

Example 1 (Non-closed asymptotic set). *This example is due to Matthieu Sablik. Consider a CA with six states: particle going to the left, particle going to the right, wall, L, R, killer. A particle makes rebounds between walls, ensuring that it is the only particle between two walls, that the cells between the left wall and itself are in state L, and that the cells between itself and the right wall are in state R. If the configuration is not well formed, a killer state appears and spreads towards both sides. Any configuration with a single particle between two walls, with L on its left and R on its right is actually periodic. If we look at a sequence of such configurations where the walls are further and further from the central cell – still containing the same particle – then it converges to the configuration where the particle is between only L on its left and only R on its right. It is easy to see that this configuration cannot be the adhering value of any orbit.*

Of course, any quiescent configuration being the limit of its own orbit, it is in the asymptotic set. Moreover, since the uniform configurations constitute a subsystem, we can see that there is always at least one uniform configuration in the asymptotic set.

The homogeneity of the CA makes the space somehow so rigid that they satisfy an analogous of Poincaré's theorem.

Theorem 3 (Bernardi [20]). *A CA is surjective if and only if its set of recurrent configurations is dense.*

In particular, from Proposition 3, a CA is surjective if and only if it is nonwandering. This fact was also proved via ergodic theory in [21]. Thus it is possible to link CA surjectivity and asymptotic set.

Corollary 26. *A CA is surjective if and only if its asymptotic set is a residual set.*

This characterization is not as strong as that we have on limit sets; one can wonder if an equivalent one could be found. Surprisingly, this simple question is still open.

Open question 1. *Does there exist a surjective CA whose asymptotic set is not full?*

If the asymptotic set is a global notion, the trace is a local observation of the behavior. In this way, it is easy to think that if some global behavior (asymptotic periodicity, asymptotic nilpotency, ...) is true, then the local observation (*i.e.*, the traces) will show the same behavior. The more interesting question is whether the converse is true or not. A first result of that kind is the following proposition.

Proposition 27. *If $p \in \mathbb{N}_+$, then a TDDS (Σ, F) is asymptotically p -periodic if and only if all of its traces $\tau_F^{(k)}$, for $k \in \mathbb{N}$, are weakly p -preperiodic.*

Proof. Let $x \in \omega_F$, *i.e.* there exists a configuration y whose orbit $\mathcal{O}_F(y)$ admits x as an adhering value. If any trace is p -preperiodic, then for any $k \in \mathbb{N}$, there exists a generation $j \in \mathbb{N}$ such that $T_F^{(k)} F^j(y)$ is p -periodic. By continuity of the trace, $T_F^{(k)}(x)$ is p -periodic. Putting things together, x is p -periodic. The converse comes from the preservation of the asymptotic periodicity by factor maps and from Proposition 23. \square

For the specific case of PCA, the homogeneity of the rule imposes every trace to be preperiodic as soon as the trace of width 1 is, and we have the following stronger statement.

Proposition 28. *If $p \in \mathbb{N}_+$, then a PCA F is asymptotically p -periodic if and only if its trace τ_F is weakly p -preperiodic.*

Nevertheless, the preperiod cannot be made uniform: the CA Min, for instance, is asymptotically periodic but not weakly preperiodic. Even imposing a unique ultimate periodic word in each trace cannot help get a bounded preperiod, as illustrated by the following example.

Example 2 (Non-preperiodic CA of weakly preperiodic trace). *Let F be the CA defined on alphabet $\{0, 1, 2\}$, with anchor 1, diameter 4 by the following local rule:*

$$\begin{aligned} \{0, 1, 2\}^4 &\rightarrow \{0, 1, 2\} \\ f : (x_{-1}, x_0, x_1, x_2) &\mapsto \begin{cases} x_{-1} + 1 \bmod 3 & \text{if } x_{-1} \neq x_0 \neq x_1 = x_2 ; \\ x_0 + 1 \bmod 3 & \text{otherwise.} \end{cases} \end{aligned}$$

By recurrence, one can see that two consecutive cells in the same state will always keep an identical state. In particular, a cell that applies the first part of the rule gets the same state as its right neighbor, and both of them will never apply the second part of the rule after that. As a conclusion, $\tau_F \subset \mathcal{O}_\sigma((012)^*(02 + 12 + 01)(^\infty 012))$ is weakly 3-preperiodic.

Let Φ the simulation by $(\Sigma_K \subset \{0, 1, 2\}^{\mathbb{Z}}, \sigma^2)$ of $(2^{\mathbb{N}}, \sigma)$ defined:

$$\text{on subshift } \Sigma_K \text{ of forbidden language } K = \bigcup_{\substack{a, b \in \{0, 1, 2\} \\ k \in \mathbb{N}}} aaA^{2k+1}bb \cup \{000, 111, 222\}$$

$$\text{by the local rule } \phi : (x_0, x_1) \mapsto \begin{cases} 1 & \text{if } x_0 \neq x_1 ; \\ 0 & \text{otherwise.} \end{cases}$$

From the definition of f , Φ is a conjugacy of $F|_{\Sigma_K}$ into Min . In particular, F simulates a CA that is not preperiodic. Therefore, neither can F be preperiodic.

Total disconnection allows a generalization of Proposition 21.

Proposition 29. *No asymptotically periodic TDDS is sensitive.*

Proof. Suppose that (Σ, F) is a sensitive asymptotically periodic TDDS. By Propositions 7 and 22, all of its traces sufficiently thin also have both properties, which contradicts Corollary 24. \square

Similarly to nilpotency and weak nilpotency, we can see, thanks to shift-invariance of the asymptotic set, that any configuration z such that some CA is asymptotically z -nilpotent is uniform. We will speak of asymptotically 0-nilpotent CA, where 0 is a quiescent state of A .

We can use Proposition 22 to deduce that a TDDS F is asymptotically nilpotent if and only if all of its traces are weakly nilpotent. This result can be simplified if F is a PCA, since each projection of $\tau_F^{(k)}$ coincides with the trace τ_F . Hence F is asymptotically nilpotent if and only if its trace τ_F is weakly nilpotent. We are going to prove that, in the case of one-dimensional CA, asymptotic nilpotency is a very strong property, equivalent to nilpotency.

Lemma 1. *Let F a PCA on some twosided SFT Σ , such that for any generation $j \in \mathbb{N}$, there exists some 0-finite 0-nilpotent configuration $x \in \Sigma$ such that $F^j(x) \neq \infty 0^\infty$. Then F is not asymptotically 0-nilpotent.*

Proof. Assume F has radius $r \in \mathbb{N}$ and is asymptotically nilpotent.

Let us first show that the configuration can be taken with “holes”, i.e. for any $k \in \mathbb{N}$, there is a 0-finite 0-nilpotent configuration $x' \in [0^{(k)}]$ and a generation $j > k$ such that $F^j(x')_0 \neq 0$. Indeed, by asymptotic nilpotency and compactness, there exists a generation $J \in \mathbb{N}$ such that $\forall x \in \Sigma, \exists j < J, F^j(x) \in [0^{(k)}]$. By hypothesis, and maybe thanks to a composition by a shift, there exists a 0-finite 0-nilpotent configuration x such that $F^{k+J}(x)_0 \neq 0$; hence there exists a configuration $x' = F^j(x) \in [0^{(k)}]$, that is still 0-finite and 0-nilpotent (as are all the configurations of the orbit of x), such that $F^{k+J-j}(x')_0 \neq 0$, with $j < J$ and hence $k + J - j > k$.

Let us now show that if $x \in \Sigma$ is a 0-finite 0-nilpotent configuration and $k \in \mathbb{N}$, then there exists a 0-finite 0-nilpotent configuration $y \in [x_{\langle rk \rangle}]$ such that

$\{j \in \mathbb{N} \mid F^j(y)_0 \neq 0\} \supsetneq \{j \in \mathbb{N} \mid F^j(x)_0 \neq 0\}$. Indeed, $F^n(x) = {}^\infty 0^\infty$ for some generation $n \in \mathbb{N}$. k can be enlarged so that we can suppose that $x \in {}^\infty 0[A^{2r(k-2n)}]^\infty 0$. The previous point gives a 0-finite 0-nilpotent configuration $x' \in [0^{(2rk)}]$ and a generation $j > k$ such that $F^j(x')_0 \neq 0$. n can be enlarged so that one can assume that Σ is a $2n$ -SFT; consequently, it contains the configuration $y = x'_{[-\infty, -rk]} [x_{(rk)}] x'_{[rk, \infty]}$. By an immediate recurrence on generation $j \leq n$, we can see that $F^j(y)_i = F^j(x')_i$ if $|i| > r(k - 2n - j)$ and $F^j(y)_i = F^j(x)_i$ if $|i| \leq r(k - 2n + j)$. In particular:

$$\{j \in \mathbb{N} \mid F^j(y)_0 \neq 0\} \cap \llbracket 0, n \rrbracket = \{j \in \mathbb{N} \mid F^j(x)_0 \neq 0\} \cap \llbracket 0, n \rrbracket = \{j \in \mathbb{N} \mid F^j(x)_0 \neq 0\}.$$

On the other hand, since $F^n(x)_{(r(k-n))} = 0^{2r(k-n)} = F^n(x')_{(r(k-n))}$, one can see that $F^n(y) = F^n(x')$. By construction, there is a generation $j \geq k > n$ such that $F^j(y)_0 = F^j(x')_0 \neq 0$. As a result, $\{j \in \mathbb{N} \mid F^j(y)_0 \neq 0\} \supsetneq \{j \in \mathbb{N} \mid F^j(x)_0 \neq 0\}$.

Therefore, we can inductively build a sequence $(y^k)_{k \in \mathbb{N}}$ of 0-finite 0-nilpotent configurations, with $x^0 = {}^\infty 0^\infty$ and for any $k \in \mathbb{N}$, $x^{k+1} \in [x^k_{(r(k+1))}]$ and $\{j \in \mathbb{N} \mid F^j(x^{k+1})_0 \neq 0\} \supsetneq \{j \in \mathbb{N} \mid F^j(x^k)_0 \neq 0\}$. This sequence converges towards the configuration $x \in \bigcap_{k \in \mathbb{N}} [x^k_{(r(k+1))}]$, which is such that $\{j \in \mathbb{N} \mid F^j(x)_0 \neq 0\}$ contains $\{j \in \mathbb{N} \mid F^j(y^k)_0 \neq 0\}$ for any $k \in \mathbb{N}$ (by continuity of the trace application). This sequence of sets being strictly increasing, $\{j \in \mathbb{N} \mid F^j(x)_0 \neq 0\}$ is infinite, *i.e.* the trace τ_F is not weakly 0-nilpotent. \square

With this lemma, we can prove the already mentioned theorem linking the global behavior and asymptotic behavior of the radius 1 traces. This generalizes a result presented in [7].

Theorem 4. *Any asymptotically 0-nilpotent PCA on a transitive SFT is 0-nilpotent.*

Proof. Let F be an asymptotically 0-nilpotent PCA of radius $r \in \mathbb{N}$ over some transitive SFT $\Sigma \subset A^{\mathbb{M}}$, whose order l can be assumed equal to $2r$ (enlarging l or r if need be). We can assume that $\mathbb{M} = \mathbb{Z}$ without altering the properties of nilpotency and limit nilpotency. Proposition 21 and Theorem 2 give an l -blocking word u , *i.e.* $T_F^l([u]_{-i})$ is a singleton for some $i \in \mathbb{Z}$. There exists some generation $k \in \mathbb{N}$ such that $\forall n \geq k, \forall x \in [u]_{-i}, F^n(x) \in [0^l]_0$. Let $j \in \mathbb{N}$. Suppose that F is not nilpotent; it gives a configuration $x \in A^{\mathbb{Z}}$ such that $F^{j+k}(x)_0 \neq 0$. Σ being transitive, it contains a configuration $x' = zuv[x_{(r(j+k))}]v'u z'$, with $z \in A^{-\mathbb{N}}, z' \in A^{\mathbb{N}}, v, v' \in A^*$. This latter configuration has the property that, if $p_1 = -r(j+k) - |uv| + i$ and $p_2 = r(j+k) + |v'| + i$, then:

$$\forall n \geq k, F^n(x')_{\llbracket p_1, p_1+l \rrbracket} = F^n(x')_{\llbracket p_2, p_2+l \rrbracket} = 0^l.$$

Σ being an l -SFT containing ${}^\infty 0^\infty$ (as the limit of all the orbits of F), it also contains the configuration $y = {}^\infty 0[F^k(x')_{\llbracket p_1, p_2+l \rrbracket}]_{p_1} {}^\infty 0$. By construction, $F^j(y)_0 = F^{j+k}(x)_0 \neq 0$. As the concatenation of parts of three configurations sharing the same traces of width l in cells p_1 and p_2 , one can see from Proposition 1 that for any generation $n \in \mathbb{N}$, $F^n(y)_{\llbracket -\infty, p_1+l \rrbracket} = {}^\infty 0$ and $F^n(y)_{\llbracket p_2, \infty \rrbracket} = 0^\infty$. Besides, asymptotic nilpotency gives a generation $n \in \mathbb{N}$ for which $F^n(y) \in [0^{p_2-p_1-l}]_{p_1+l}$; it results that $F^n(y) = {}^\infty 0^\infty$.

The configuration y is 0-finite, 0-nilpotent, but dies arbitrarily late (after at least j generations); this contradicts Lemma 1. \square

Note that the juxtaposition of the blocking words is, as in the proof of Theorem 2, the crucial point that prevents a direct generalization of the proof to higher dimensions, that would nevertheless seem natural (CA can be defined similarly on any grid).

Conjecture 1. *Any asymptotically nilpotent d -dimensional CA $(A^{\mathbb{M}^d}, F)$, $d \in \mathbb{N}_+$, is nilpotent.*

In the case of the limit set, there was a clear dichotomy between nilpotency (the limit set is a singleton) and other cases (the limit set is infinite). Here, we cannot achieve such a dichotomy (think about the Min automaton which has an asymptotic set with only two singletons). However, it is possible to achieve a similar result looking at finite configurations. Let us introduce variants up to a shift of already introduced notions: an (F, σ) -periodic configuration is a $F\sigma^k$ -periodic configuration for some $k \in \mathbb{Z}$. A jointly F -periodic configuration is an F -periodic and σ -periodic configuration. Let \mathcal{J}_F be the set of jointly F -periodic configurations.

If F is a (twosided) PCA of radius r over some subshift $\Sigma \subset A^{\mathbb{Z}}$, and $0 \in A$, we say that a configuration x is $(F, 0)$ -separated with width $k \in \mathbb{N}_+$, time $J \in \mathbb{N}_+$ and shift $s \in \llbracket -rJ, rJ \rrbracket$ if $x_{\llbracket -2rJ, 0 \rrbracket} = x_{\llbracket k, k+2rJ \rrbracket} = 0^{2rJ}$ and $F^J(x)_{\llbracket -rJ, k+rJ \rrbracket} = x_{\llbracket s-rJ, s+k+rJ \rrbracket} \neq 0^{k+2rJ}$. To state our result on the form of the asymptotic set of non-nilpotent PCA, we previously need to prove the following lemmata: the first one uses the finite type condition to build new configurations inside the asymptotic set from some known specific ones, and the last one ensures that this is applicable in any non-nilpotent PCA.

Lemma 2. *Let F be a (twosided) PCA of radius $r > 0$ on some SFT $\Sigma \subset A^{\mathbb{Z}}$ of order r , containing some $(F, 0)$ -separated configuration. Then Σ contains some nonuniform jointly periodic configuration.*

Proof. Let $x \in \Sigma$ be $(F, 0)$ -separated with width $k \in \mathbb{N}_+$, time $J \in \mathbb{N}_+$ and shift $s \in \llbracket -rJ, rJ \rrbracket$. Let $y \in A^{\mathbb{Z}}$ be the configuration of period $k + 2rJ$ such that $y_{\llbracket 0, k+2rJ \rrbracket} = x_{\llbracket 0, k+2rJ \rrbracket}$. Thanks to the separation, each pattern of width $2rJ$ appearing in y also appears in x :

$$\forall i \in \mathbb{N}, y_{\llbracket i, i+2rJ \rrbracket} = x_{\llbracket i \bmod (k+2rJ), i \bmod (k+2rJ)+2rJ \rrbracket}.$$

It results that y is in the SFT Σ of order r . Moreover, again by catenating patterns of width $2rJ$ from the definition, we can see that $F^J \sigma^s(y) = y$. In particular, $F^{(k+2rJ)J} \sigma^{(k+2rJ)s}(y) = y$, but we also know by construction that $\sigma^{(k+2rJ)s}(y) = y$, hence y is jointly periodic. \square

We say that a subshift is nontrivial if it is not reduced to a single configuration.

Lemma 3. *Let F be a surjective twosided PCA on some nontrivial subshift $\Sigma \in A^{\mathbb{Z}}$. Then Σ contains either some $(F, 0)$ -separated configuration or some 0-infinite configuration.*

Proof. If 0 is not quiescent, then Σ contains some uniform configuration distinct from $\infty 0 \infty$. Otherwise, by surjectivity and nontriviality, there exist $(x^j)_{j \in \mathbb{Z}} \in \Sigma^{\mathbb{Z}}$ and $(s_j)_{j \in \mathbb{Z}}$ with for any $j \in \mathbb{Z}$, $|s_j - s_{j-1}| \leq r$, $x^{j+1} = F(x^j)$, and $\sigma^{s_j}(x^j)_0 \neq 0$. Let $t_1 = 1$, $l_1 = 0$ and, for $k \in \mathbb{N}_+$, $t_{k+1} = (|A|^{2l_k} + 1)t_k$ and $l_{k+1} = l_k + 4rt_{k+1}$. For $k \in \mathbb{N}_+$, define $U_k = \{y \in \Sigma \mid y_{[-l_k-2rt_{k+1}, -l_k]} \neq 0^{2rt_{k+1}} \text{ or } y_{[l_k, l_k+2rt_{k+1}]} \neq 0^{2rt_{k+1}}\}$. Let us prove by recurrence on $k \in \mathbb{N}_+$ that $\forall J \in \mathbb{Z}, \exists j \in \llbracket J - t_k, J \rrbracket, \sigma^{s_j}(x^j) \in \bigcap_{1 \leq l < k} U_l$.

The case $k = 1$ is trivial (no intersection).

Assume that $k \in \mathbb{N}_+$ is such that $\forall J \in \mathbb{Z}, \exists j \in \llbracket J - t_k, J \rrbracket, \sigma^{s_j}(x^j) \in \bigcap_{l < k} U_l$. Applying this property $|A|^{2l_k} + 1$ times, for any $J \in \mathbb{Z}$, we can find some distinct $(j_m)_{0 \leq m \leq |A|^{2l_k}}$ in $\llbracket J - t_{k+1}, J \rrbracket$ such that $\sigma^{s_{j_m}}(x^{j_m}) \in \bigcap_{l < k} U_l$ for any $m \in \llbracket 0, |A|^{2l_k} \rrbracket$. Suppose that for any $m \in \llbracket 0, |A|^{2l_k} \rrbracket$, $\sigma^{s_{j_m}}(x^{j_m}) \notin U_k$, i.e. $\sigma^{s_{j_m}}(x^{j_m})_{[-l_k-2rt_{k+1}, -l_k]} = \sigma^{s_{j_m}}(x^{j_m})_{[l_k, l_k+2rt_{k+1}]} = 0^{2rt_{k+1}}$. By the pigeon-hole principle, there are some $m, m' \in \llbracket 0, |A|^{2l_k} \rrbracket$ with $m < m'$ such that $\sigma^{s_{j_m}}(x^{j_m})_{[-l_k, l_k]} = \sigma^{s_{j_{m'}}}(x^{j_{m'}})_{[-l_k, l_k]}$. Hence $\sigma^{s_{j_m} - l_k}(x^{j_m})$ is $(F, 0)$ -separated with width $2l_k \in \mathbb{N}_+$, time $j_{m'} - j_m \leq J$ and shift $s_{j_{m'}} - s_{j_m}$.

We have just proved that if there are no $(F, 0)$ -separated configurations, then for any $k \in \mathbb{N}_+$, $\forall J \in \mathbb{Z}, \exists j \in \llbracket J - t_k, J \rrbracket, \sigma^{s_j}(x^j) \in \bigcap_{l < k} U_l$.

By compactness, the closed intersection $\bigcap_{l \in \mathbb{N}} U_l$ is nonempty. By definition it contains some 0-infinite configuration. \square

Proposition 30. *Let F be a (twosided) PCA over some SFT $\Sigma \subset A^{\mathbb{Z}}$ and $\Lambda \supset \mathcal{J}_F$ a nontrivial strongly F -invariant subshift of Σ . Then Λ contains some 0-infinite configuration.*

Proof. (Λ, F) is surjective, hence Lemma 3 gives either some 0-infinite configuration or some $(F, 0)$ -separated configuration. But if (Σ, F) admits an $(F, 0)$ -separated configuration, then Lemma 2 gives a nonuniform jointly periodic configuration, which is clearly infinite, and belongs in Λ by hypothesis. \square

This proposition can be applied in particular to the closure of ω_F , which satisfies all the hypotheses, as previously stated.

Corollary 31. *A (twosided) CA F over $A^{\mathbb{Z}}$ is 0-nilpotent if and only if $\overline{\omega_F}$ (resp. Ω_F) contains only 0-finite configurations.*

Unlike the limit set, the asymptotic set of CA is very sensitive to shift compositions. Actually, shifting sufficiently a CA allows any limit configuration to become the adhering values of some orbit. We say that a DDS (X, F) is *semitransitive towards* $Y \subset X$ if for any nonempty open set $U \subset X$, any ball $\mathcal{B}_\varepsilon(x)$ of center $x \in Y$ and radius $\varepsilon > 0$, and any generation $J \in \mathbb{N}$, there exists $j \geq J$ such that $F^j(U) \cap \mathcal{B}_\varepsilon(x) \neq \emptyset$. We can now generalize Proposition 18 with this notion.

Lemma 4. *Let (X, F) some DDS. The asymptotic set ω_F includes all the subsets $Y \subset X$ such that there is a subsystem (X', F) which is semitransitive towards Y .*

Proof. Let x a point of such a subset Y , and $U_0 = X'$. By induction, semitransitivity gives us sequences $(j_k)_{k \in \mathbb{N}_+}$ of integers and $(U_k)_{k \in \mathbb{N}_+}$ of open sets of X' such that $j_k > k$, $F^{j_k}(U_k) \cap \mathcal{B}_{1/k}(x) \neq \emptyset$ and $U_{k+1} = U_k \cap F^{-j_k}(\mathcal{B}_{1/k}(x))$. By compactness, the intersection $\bigcap_{k \in \mathbb{N}} U_k$ is nonempty, and any of its elements admits x as orbit adhering value. \square

Proposition 32. *Let F be some oblic CA. Then $\omega_F = \Omega_F$.*

Proof. By Lemma 4, it is enough to show that $(A^{\mathbb{Z}}, F)$ is semitransitive towards Ω_F . Let $k, l \in \mathbb{N}$, $u \in A^{(k)}$, and x a configuration of the limit set, *i.e.* for any generation $j \in \mathbb{N}$, there exists a configuration x^j such that $F^j(x^j) = x$. Assume F has an anchor $m < 0$, a diameter $d \in \mathbb{N}$, a local rule f , and that $k+1+l = jm$ for some $j \in \mathbb{N}$ (otherwise enlarge one of them). It is then easy to see that any configuration $y \in [ux^j]_{k,l+(d-1-m)j}]_{-k}$ is in the open set $U = [u]$ and satisfies $F^j(y) \in [x]_{[-l,l]} = \mathcal{B}_{2^{-l}}(x)$. \square

Any local rule can be seen as that of an oblic CA; this brings the following restatement.

Corollary 33. *If F is a CA of anchor $m \in \mathbb{Z}$ and anticipation $m' \in \mathbb{N}$, then $\Omega_F = \omega_{\sigma^k F}$ for any $k > m$ and any $k < -m'$.*

The previous statements allow smart constructions of some counter-examples.

Example 3 (Asymptotic set strictly including the union of the transitive subsystems). *The CA $\sigma \text{ Min}$ has anchor -1 , diameter 2 and the same local rule as the Min CA, *i.e.* :*

$$f : \begin{array}{ccc} \mathbb{2}^2 & \rightarrow & \mathbb{2} \\ (a, b) & \mapsto & a \times b \end{array}.$$

As an oblic CA, its asymptotic set is equal to its limit set: $\omega_{\sigma \text{ Min}} = \Omega_{\sigma \text{ Min}} = (\infty 0 + \infty 1)1^(0^\infty + 1^\infty)$. It includes strictly the asymptotic set of the limit system: $\omega_{\sigma \text{ Min}|_\Omega} = \omega_{\sigma \text{ Min}|_\omega} = \omega_{\text{Min}} = \{\infty 0^\infty, \infty 1^\infty\}$, which is also the union of the transitive subsystems.*

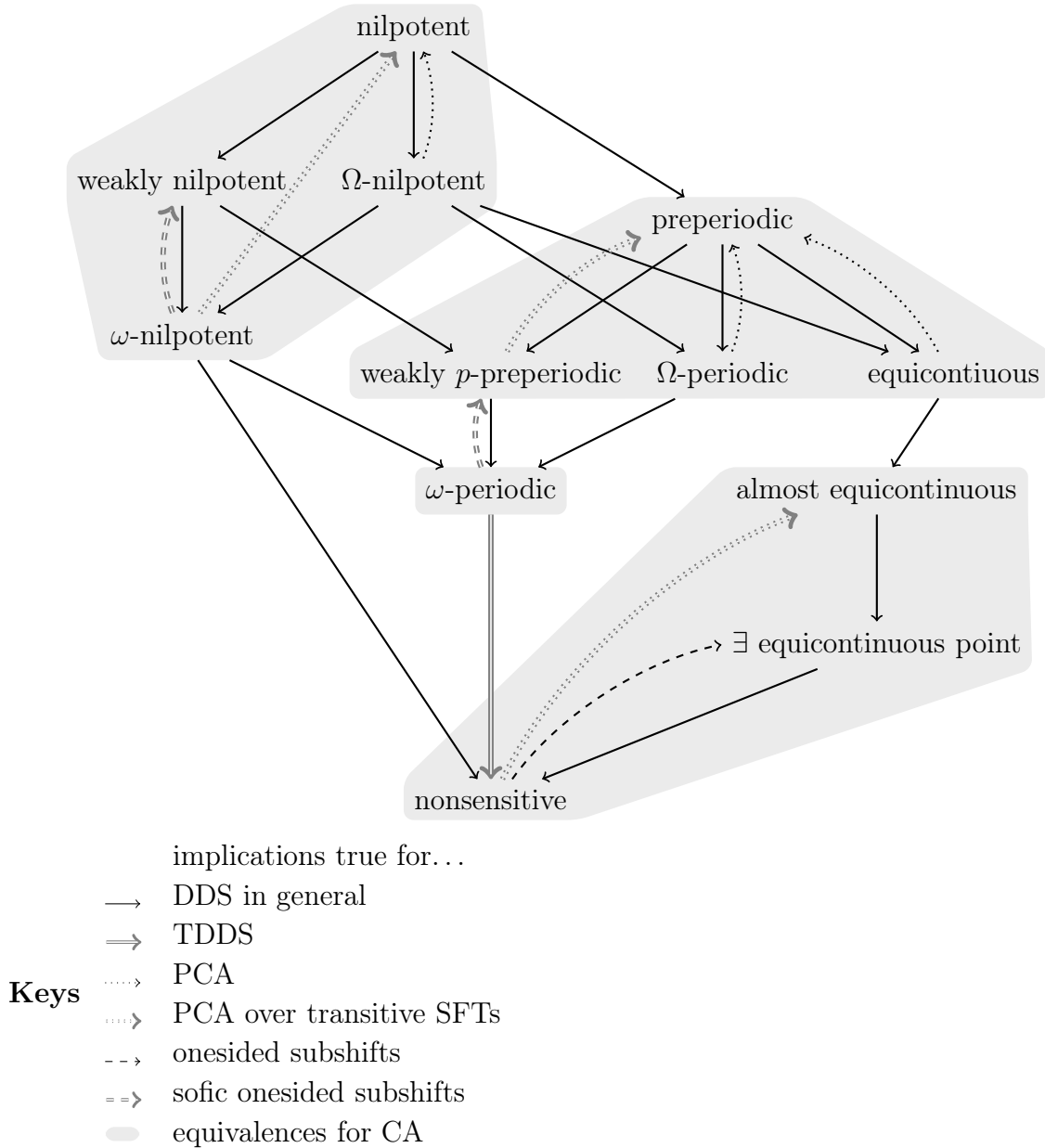
This example brings the following questions: is $\omega_{F|_\omega}$ always the union of the transitive subsystems? How can we understand the fact that these two sets only differ by isolated configurations (the Cantor-Bendixon derivative)? When we look at the action of the shift over the asymptotic set, do we always have, as in the Min case, a “minimum” asymptotic set, which corresponds to the asymptotic set of the limit system of all the shifted versions?

4. Conclusion

We studied discrete-time dynamical systems with respect to their behavior in the (very) long term. After several general remarks on the topological properties of their limit set and asymptotic set, we focused on particular systems: subshifts (especially sofic subshifts) and partial cellular automata (especially cellular automata). In these

two cases, the limit set and the asymptotic set have a very specific structure. The homogeneity of the models makes the properties over the limit behavior to constrain the possible transient evolution.

The diagram below summarizes the main implications we proved (note that Ω -nilpotent, ω -nilpotent, Ω -periodic and ω -periodic stand respectively for limit-nilpotent, asymptotically nilpotent, limit-periodic and asymptotically periodic).



Acknowledgments

This work was partially supported by the ANR project *EMC* and the Academy of Finland project 131558. We would also like to thank Guillaume Theyssier, Mathieu Sablik, François Blanchard, for very interesting discussions about the asymptotic set.

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